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On the irregularity of cyclic coverings of the projective plane

Fumio Sakai

(Preliminary Version)

1 Introduction

The aim of this note is to give a survey on the irregularity of cyclic coverings of the projective plane $\mathbf{P}^2$. Let $f(x,y)$ be a polynomial of degree d over $\mathbf{C}$. Let us consider the cyclic multiple plane:

$$z^n = f(x,y).$$

Decompose f into irreducible components: $f = f_1^{m_1}\cdots f_r^{m_r}$. We assume that the condition: $\mathrm{GCD}(n, m_1, ..., m_r) = 1$ is satisfied. This is nothing but the condition that the above surface is irreducible. We pass to the projecteve model. Let $\tilde{f}(x_0, x_1, x_2)$ be the homogeneous poynomial associated to f so that $\tilde{f}(1,x,y) = f(x,y)$. Let C be the plane curve defined by the equation: $\tilde{f} = 0$. Let C_i be the irreducible component $\tilde{f}_i = 0$. Let L denote the infinite line: $x_0 = 0$. Define e to be the smallest integer with the condition: $e \geq n/d$. Set $m_0 = ne - d$. Note that $m_0 = 0$ if and only if n divides d. Let W_n be the normalization of the following weighted hypersurface in $\mathbf{P}(1,1,1,e)$:

$$x_3^n = x_0^{m_0}\tilde{f}(x_0, x_1, x_2).$$

The covering map $W_n \to \mathbf{P}^2$ ramifies over C in case $m_0 = 0$ or over $C \cup L$ in case $m_0 \neq 0$. Let $\pi : X_n \to W_n$ be a desingularization. Let $\varphi : X_n \to \mathbf{P}^2$ be the composed map.

Definition. The irregularity $q(X_n)$ of X_n has three equivalent expressions:

$$q(X_n) = \dim H^1(X_n, \mathcal{O}) = \dim H^0(X_n, \Omega^1) = \frac{1}{2}\dim H^1(X_n, \mathbf{R})$$

There are four classical references on this topics:de Franchis [dF], Comessatti [C], Zariski [Z1], [Z2]. My personal motivation to this question is its application to the analysis of singular plane curves. Cf. [S].

Proposition 1 (Easy Bound).

$$2q(X_n) \leq \sum_{i=0}^{r} d_i(n - n_i) - 2(n-1)$$

where $n_i = \mathrm{GCD}(n, m_i)$, $d_i = \deg f_i$ *and* $d_0 = 1$. *Note that* $n_0 = \mathrm{GCD}(n,d)$.

Proof. Let $\Gamma \in X_n$ be the inverse image of a general line on $\mathbf{P}^2$. We can easily prove that $H^1(X_n, \mathcal{O})$ injects to $H^1(\Gamma, \mathcal{O})$. The Hurwitz formula gives the genus of Γ.

Corollary.

$$2q(X_n) \leq \begin{cases} (n-1)(\sum_{i=1}^r d_i - 2) & \text{if } n|d \\ (n-1)(\sum_{i=1}^r d_i - 1) & \text{otherwise} \end{cases}$$

Let us exibit examples with positive irregularity. Let $\Gamma_k \to \mathbf{P}^1$ be a k-fold cyclic covering. Given a rational map $\phi : \mathbf{P}^2 \to \mathbf{P}^1$. Suppose that Γ_k is given by the equation:$y_2^k = \prod(b_i y_0 - a_i y_1)^{\ell_i}$ (we may assume $k|\sum \ell_i$) and that the map ϕ is given by $(G(x_0, x_1, x_2), H(x_0, x_1, x_2))$ where both G and H are homogeneous polynomials of degree ℓ. If $n|\ell \cdot \sum \ell_i$ and $k|n$, then the multiple plane X_n defined by the equation: $x_3^n = \prod(b_i G(x) - a_i H(x))$ factors through Γ_k. In this case, we say that X_n *factors through a pencil*. We see that X_n has positive irregularity if Γ_k has positive genus.

In order to investigate the irregularity of cyclic coverings of $\mathbf{P}^2$, there are three approaches: (i) through the behavior of rational differential forms, cf. Esnault [E], Zuo [Z] (ii) through the action of the cyclic group $\mathbf{Z}_n$ on the Albanese variety, cf. Khashin [K], Catanese-Ciliberto [CC] (iii) through the topology of complements of the branch curves. cf. Libgober [L], Randell [R], Kohno [Ko], Loeser-Vaquié [LV], Dimca [D].

2 Differential forms

Let $\psi : S \to \mathbf{P}^2$ be a composition of blow-ups so that the inverse image of $C \cup L$ has normal crossings. Write

$$\psi^*(\sum_{i=0}^r m_i C_i) = \sum \nu_j D_j.$$

Here we set $C_0 = L$. We understand that if $j \leq r$, D_j is the strict transform of C_j and $\nu_j = m_j$ and that for $j > r$, D_j is exceptional for ψ. Since $\psi^*(\sum_{i=0}^r m_i C_i) \in |n\psi^* \mathcal{O}(e)|$, one can construct an n-fold covering of S, which ramifies over $\psi^*(\sum_{i=0}^r m_i C_i)$. Let W_n' denote its normalization. Up to birational equivalence, we have the commutative diagram:

$$\begin{array}{ccccc} W_n & \leftarrow & W_n' & \leftarrow & X_n \\ \downarrow & & \downarrow & \swarrow \phi & \\ \mathbf{P}^2 & \leftarrow & S & & \end{array}$$

Set $\zeta = e^{2\pi i/n}$. The eigenspace decomposition of the structure sheaf $\mathcal{O}_{X_n}$ has the following consequence:

Proposition 2 (Esnault [E]). *In this situation, we have*

$$H^0(X_n, \mathcal{O}_{X_n})^{\zeta^i} \cong H^0(S, \mathcal{L}^{(i)^{-1}}),$$

where $\mathcal{L}^{(i)} = \psi^* \mathcal{O}(ie) \otimes \mathcal{O}(-\sum[i\nu_j/n] D_j)$.

As for the eigenspace decomposition of the sheaf Ω^1, we have

Proposition 3 ([E], [Zu]). *One has*

$$H^0(X_n, \Omega^1)^{\zeta^i} \cong H^0(S, \Omega^1(\log D(i)) \otimes \mathcal{L}^{(i)^{-1}}),$$

where $D(i) = \sum(i\nu_j - n[i\nu_j/n])D_j$.

Remark. Note that $D_j \not\subset D(i)$ if and only if $n|i\nu_j$.

The Bogomolov type vanishing theorem gives the following criterion for the vanishing of the irregularity.

Theorem 1 ([E],[Zu]). *If $D(i)$ is big for all i, then $q(X_n) = 0$.*

Proof. If $H^0(X_n, \Omega^1)^{\zeta^i} \neq 0$, then one finds that $\mathcal{L}^{(i)} \hookrightarrow \Omega^1(\log D(i))$, which is impossible if $D(i)$ is big, since $D(i) \in |(\mathcal{L}^{(i)})^{\otimes n}|$.

Since $q(X) = p_g(X) + 1 - \chi(\mathcal{O})$, one gets the irregularity $q(X_n)$ if one knows $p_g(X_n)$ and $\chi(\mathcal{O}_{X_n})$.

Proposition 4.

$$H^0(X_n, \Omega^2)^{\zeta^i} \cong H^0(S, \Omega^2(\log D(i)) \otimes \mathcal{L}^{(i)^{-1}}).$$

On the other hand, one has the following formula for the term $\chi(\mathcal{O})$.

Proposition 5.

$$\chi(\mathcal{O}_{X_n}) = \sum_{i=0}^{n-1} \chi(\mathcal{O}_{\mathbf{P}^2}(-(ie - \sum[i\nu_j/n]d_j))) - \dim R^1\pi_*\mathcal{O}_{X_n}.$$

Proof. Taking the direct image sheaf, we see that

$$\chi(\mathcal{O}_{X_n}) = \chi((\psi \circ \phi)_*\mathcal{O}_{X_n}) - \dim R^1(\psi \circ \phi)_*\mathcal{O}_{X_n}.$$

We have

$$(\psi \circ \phi)_*\mathcal{O}_{X_n} \cong \psi_*(\mathcal{L}^{(i)^{-1}}) \cong \mathcal{O}(-(ie - \sum[i\nu_j/n]d_j)),$$

and

$$\dim R^1(\psi \circ \phi)_*\mathcal{O}_{X_n} = \dim R^1\pi_*\mathcal{O}_{X_n}.$$

Problem. *Discuss those line arrangements C such that X_n has positive irregularity for some n.*

3 Albanese map

Let X_n be a non-singular model of a cylic multiple plane as defined in Introduction. We denote by G the cyclic group $\mathbf{Z}_n$ and let σ be its generator. Suppose $q(X_n) > 0$. We have the Albanese map $\alpha : X_n \rightarrow \mathrm{Alb}(X_n)$. The group G acts on X_n and naturally on $\mathrm{Alb}(X_n)$.

Proposition 6. *If the Albanese image $\alpha(X_n)$ is a curve, then X_n factors through a pencil.*

Proof. Set $\Gamma = \alpha(X_n)$. The group also acts on Γ. We infer that Γ/G is isomorphic to $\mathbf{P}^1$, because there exists a rational map from $\mathbf{P}^2$ onto it.

Proposition 7. *Suppose that there exist two liniearly independent holomorphic one forms ω, ω' such that $\sigma^*\omega = \lambda\omega$, $\sigma^*\omega' = \lambda^{-1}\omega'$ for some λ. Then the Albanese image $\alpha(X_n)$ is a curve.*

Proof. By hypothesis, we find that $\sigma^*(\omega \wedge \omega') = \omega \wedge \omega'$. So $\omega \wedge \omega'$ must be a pull-back of a holomorphic 2-form on $\mathbf{P}^2$, hence $\omega \wedge \omega' = 0$. The assertion follows from the Castelnuovo-de Franchis theorem.

Proposition 8. *Suppose that there exists an n-th root of unity λ ($\lambda \neq \pm 1$) such that $\sigma^*\omega = \lambda\omega$ for all $\omega \in H^0(X_n, \Omega^1)$. Then λ can take one of the values $\pm i$, $\pm\rho$, $\pm\rho^2$ where $\rho = e^{2\pi i/3}$. Furthermore,*

$$\mathrm{Alb}(X_n) \cong E_\lambda^q,$$

where E_λ is the elliptic curve $\mathbf{C}/\mathbf{Z} \oplus \mathbf{Z}\lambda$.

Proof. Cf. Comessatti [C].

Theorem 2 (de Franchis [dF]). *If $q(X_2) > 0$, then X_2 factors through a pencil.*

Proof. In case $n = 2$, one must have $\sigma^*\omega = -\omega$ for all $\omega \in H^0(X_n, \Omega^1)$. So the assertion follows from Propositions 6 and 7.

Theorem 3 (Comessatti [C]). *If $q(X_3) > 0$ and if the Albanese image of X_3 is a surface, then $\mathrm{Alb}(X_3) \cong E_\rho^q$.*

Proof. This follows from Propositions 7 and 8.

We can prove this type of results for the cases $n = 4, 6$, which were also proved by Catanese and Ciliberto [CC].

Theorem 4. *If $q(X_4) > 0$, then either X_4 factors through a pencil, or $\mathrm{Alb}(X_4) \cong E_i^q$.*

Proof. If $H^0(X_4, \Omega^1)^{(-1)} \neq 0$, then the surface: $x_3^2 = x_0^{m_0}\tilde{f}$ factors through a pencil, so does X_4. In case $H^0(X_4, \Omega^1)^{(-1)} = 0$, by Propositions 7 and 8, we see that either X_4 factors through a pencil or $\mathrm{Alb}(X_4) \cong E_i^q$.

Example. $z^4 = (y^2 - 2x^3)x^2(x^2+1)^2(y+2x)$. *In this case,* $X_4 \cong E_i^2$.

4 Alexander polynomials

Set

$$U = \mathbf{C}^2 \backslash \{f = 0\} = \mathbf{P}^2 \backslash C \cup L.$$

Write $U_n = \varphi^{-1}(U) \subset X_n$. We see that $\varphi : U_n \to U$ is an unramified covering of degree n. We have a commutative diagram:

$$\begin{array}{ccc} U_n & \xrightarrow{\varphi} & U \\ \downarrow & & \downarrow f \\ \mathbf{C}^* & \ni z \to z^n \in & \mathbf{C}^* \end{array}$$

The idea of the topological approach is to calculate the first Betti number of X_n through that of U_n. Namely, we write:

$$b_1(X_n) = b_1(U_n) - B.C.$$

The term $B.C.$ (the boundary contribution) is given by the following:

Proposition 9. *We have*

$$B.C. = \#\{\textit{the irreducible components of } \varphi^{-1}(C \cup L)\} - 1.$$

This follows from the following:

Proposition 10. *Let S be a smooth projective surface and let $D = D_1 \cup \ldots \cup D_n$ be a divisor having simple normal crossings. Then*

$$b_1(S) = b_1(S \backslash D) - (n - \rho(D)),$$

where $\rho(D) = \dim \{\sum \mathbf{R}[D_i]\}$ in $NS(S) \otimes \mathbf{R}$.

Proof (Esnault [E]), cf. [He]). One can deduce this from the Residue sequence:

$$0 \to H^0(S, \Omega^1) \to H^0(S, \Omega^1(\log D)) \to H^0(\hat{D}, \mathcal{O}) \to H^1(S, \Omega^1)$$

Corollary. $B.C. \geq r$.

Example. If f is reduced and if L meets C transverselly, then $B.C. = r$. Cf. [L].

One can construct an infinite cyclic covering $\tilde{U}$ of U as follows.

$$\begin{array}{ccc} \tilde{U} & \xrightarrow{\Phi} & U \\ f_\infty \downarrow & & \downarrow f \\ \mathbf{C} & \ni \tau \to e^{2\pi i \tau} \in & \mathbf{C}^* \end{array}$$

It is well known that $H_1(U, \mathbf{Z}) = \mathbf{Z}^r$, which is generated by the meridian loops γ_i around C_i. The map $f_* : \pi_1(U) \to \pi_1(\mathbf{C}^*) = \mathbf{Z}$ factors through $H_1(U, \mathbf{Z})$ and it sends

$[\gamma_1]^{s_1}\cdots[\gamma_r]^{s_r}$ to $\sum m_i s_i$. It turns out that $\tilde{U}$ is nothing but the quotient of the universal covering of U by the kernel of the above homomorphism.

Let T be the deck transformation on $\tilde{U}$ corresponding to the above infinite cyclic covering. The transformation T induces a linear transformation $T_* : H_1(\tilde{U}) \to H_1(\tilde{U})$. We have the exact sequences ([M2]):

$$\longrightarrow H_1(\tilde{U}) \xrightarrow{T_*-I} H_1(\tilde{U}) \longrightarrow H_1(U) \longrightarrow$$

Since $H_1(U, \mathbf{Z}) = \mathbf{Z}^r$, we infer that the sequence:

$$H_1(\tilde{U}, \mathbf{Z})_0 \xrightarrow{T_*-I} H_1(\tilde{U}, \mathbf{Z})_0 \to \mathbf{Z}^{r-1} \to 0 \qquad (1)$$

is exact, where $H_1(\tilde{U}, \mathbf{Z})_0 = H_1(\tilde{U}, \mathbf{Z})/\mathrm{Tor}$.

Definition. Under the assumption that $H_1(\tilde{U}, \mathbf{C})$ is finite dimensional, the *Alexander polynomial* of f is defined as follows (cf. [L]):

$$\Delta_f(t) = \det(tI - T_*).$$

Since T_* is defined on $H_1(\tilde{U}, \mathbf{Z})_0$, we infer that $\Delta_f(t) \in \mathbf{Z}[t]$. It follows from (1) that $\Delta_f(t) = (t-1)^{(r-1)} \cdot G(t)$ but $G(1) \neq 0$.

Example. Suppose that $f(x,y)$ is weighted homogeneous. Let (a,b) be the weights of (x,y) and let N be the degree of f as a weighted homogeneous polynomial. Then $U \to \mathbf{C}^*$ is a fibre bundle, of which fibre is $F = \{(x,y) | f(x,y) = 1\}$. Set $\xi = e^{2\pi i/N}$. Let $h : F \ni (x,y) \to (\xi^a x, \xi^b y) \in F$ be the monodromy map and we denote by h_* the induced linear map on $H_1(F, \mathbf{C})$. In this case, $H_1(\tilde{U}) \cong H_1(F)$ and $\Delta_f(t) = \det(tI - h_*)$. Clearly, the origin p is the only singularity of the affine curve $f = 0$ and $\det(tI - h_*)$ is known to be the local Alexander polynomial $\Delta_p(t)$ of p [M1].

Definition. In case $N = \dim H_1(\tilde{U}, \mathbf{C}) < \infty$, let $e_j(t)$, $j = 1, ..., N$, be the elementary divisors of $tI - T_*$. Set

$$N(n, T_*) = \sum \#\{\text{distinct n-th roots of unity which are roots of } e_j(t)\}.$$

Theorem 5. *If dim $H_1(\tilde{U}, \mathbf{C}) < \infty$, then*

$$2q(X_n) = 1 + N(n, T_*) - B.C..$$

Proof. We have the following exact sequence (cf. [SS]):

$$\longrightarrow H_1(\tilde{U}) \xrightarrow{T_*^n-I} H_1(\tilde{U}) \longrightarrow H_1(U_n) \longrightarrow .$$

We infer from this that $b_1(U_n) = 1 + \dim \mathrm{Ker}\,(T_*^n - I)$. We see easily that $N(n, T_*) = \dim \mathrm{Ker}\,(T_*^n - I)$.

Corollary. *If T_* is of finite order, then*

$$2q(X_n) = 1 + \#\{n\text{-th roots of unity which are roots of } \Delta_f(t)\} - B.C.$$

Definition. We say that f is *primitive* if the general fibre $f^{-1}(a)$ is irreducible. It is well known that if f is not primitive, then there are polynomials u and g such that $f(x,y) = u(g(x,y))$. Cf. [Su].

Remark. Suppose that $r \geq 2$. If f is not primitive, then (i) X_n factors through a pencil, (ii) the infinite line L does not meet C transversely.

Proposition 11. *The vector space $H_1(\tilde{U}, \mathbf{C})$ is finite dimesional if and only if either (i) $r = 1$, or (ii) $r \geq 2$, f is primitive.*

Proof. Suppose that f is primitive. The general fibre of the fibration $f_\infty : \tilde{U} \to \mathbf{C}$ is irreducible. By Lemma 7 in [Su], we see that dim $H_1(\tilde{U}, \mathbf{C}) \leq$ dim $H_1(\text{a general fibre}, \mathbf{C}) < \infty$. Note that $f_\infty^{-1}(\tau) = f^{-1}(e^{2\pi i \tau})$. Assume now that f is not primitive. Writing $f = u(g)$ as above, we set $u^{-1}(0) = \{a_1, ..., a_s\}$. Define $V = \mathbf{C}\backslash\{a_1, ..., a_s\}$. We have the diagram:

$$\begin{array}{ccc}
\tilde{U} & \rightarrow & U \\
\downarrow & & \downarrow g \\
\tilde{V} & \rightarrow & V \\
\downarrow & & \downarrow u \\
\mathbf{C} & \rightarrow & \mathbf{C}^*
\end{array}$$

If $s \geq 2$, it is easy to prove that dim $H_1(\tilde{V}, \mathbf{C}) = \infty$. It follows that dim $H_1(\tilde{U}, \mathbf{C}) = \infty$. If $s = 1$, then $\tilde{V} = \mathbf{C}$ and so dim $H_1(\tilde{U}, \mathbf{C}) < \infty$.

Remark. In case $r = 1$, this fact was pointed out in [L].

Now we come to Zariski's result.

Theorem 6 (Zariski [Z1]). *Suppose $r = 1$. If $n = p^a$ (p is a prime number), then $q(X_n) = 0$.*

Proof. Since $r = 1$, we infer from (1) that $\Delta_f(1) = \det\,(I - \tilde{h}_*) = \pm 1$. If a primitive p^i-th root of unity ($1 \leq i \leq a$) is a root of the integral polynomial $\Delta_f(t)$, then $\Delta_f(t)$ must be divided by the cyclotomic polynomial $\Phi_{p^i}(t)$. Since $\Phi_{p^i}(1) = p$, this is impossible.

We can generalize this result to the case in which C is reducible.

Theorem 7. *Suppose $r \geq 2$. Assume that f is primitive or that $n|d$. If $n = p^a$ (p is a prime number), then*

$$2q(X_n) \leq (n-1)(r-1).$$

Proof. Assume first that f is primitive. By Proposition 11, $N = \dim H_1(\tilde{U}, \mathbf{C}) < \infty$. Let $d_j(t)$ (resp. d_j) be the GCD of all j-minors of the matrix $tI - T_*$ (resp. $I - T_*$). By the exact sequence (1), we see that the elementary divisors of $I - T_*$ are $1, ..., 1, \overbrace{0, ...0}^{r-1}$. We infer that $d_j = 1$ for $j \leq N - (r-1)$ and $d_j = 0$ for $j > N - (r-1)$. Since $d_j(1) | d_j$, we find that $d_j(1) = \pm 1$ for $j \leq N - (r-1)$ and $d_j(1) = 0$ for $j > N - (r-1)$. As in the proof of Theorem 6, any primitive p^i-th root of unity other than 1 cannot be a root of $d_j(t)$ for $j \leq N - (r-1)$. Let $e_1(t), ..., e_N(t)$ be the elementary divisors of $tI - T_*$. We know that $d_j(t) = b_j e_1(t) \cdots e_j(t)$, $b_j \in \mathbf{Q}$. Thus any primitive p^i-th root of unity other than 1 cannot be a root of $e_j(t)$ for $j \leq N - (r-1)$. It follows that $N(n, T_*) \leq n(r-1)$. Since $B.C. \geq r$, we conclude that $b_1(X_n) \leq (n-1)(r-1)$.

In case $n | d$, since the infinite line L does not appear in the branch locus of $X_n \to \mathbf{P}^2$, by taking a suitable line as the infinite line, we may assume that f is primitive.

Corollary. *If $n = 2$, $r = 2$ and d is even, then $q(X_2) = 0$.*

Definition. Set $\tilde{F} = \{(x_0, x_1, x_2) \in \mathbf{C}^3 | \tilde{f}(x_0, x_1, x_2) = 1\}$. Since $\tilde{f}$ is homogeneous, $\tilde{f} : \mathbf{C}^3 \backslash \{\tilde{f} = 0\} \to \mathbf{C}^*$ is a fibre bundle. The typical fibre is $\tilde{F}$. Letting $\eta = e^{2\pi i/d}$, we have the *monodromy transformation* $\tilde{h} : \tilde{F} \ni (x_0, x_1, x_2) \longrightarrow (\eta x_0, \eta x_1, \eta x_2) \in \tilde{F}$. It induces a linear transformation $\tilde{h}_* : H_1(\tilde{F}, \mathbf{Z}) \to H_1(\tilde{F}, \mathbf{Z})$. Define

$$\Delta_C(t) = \det\, (tI - \tilde{h}_*) \in \mathbf{Z}[t],$$

which is called the *Alexander polynomial* of the plance curve C. Cf. [R], [D].

Proposition 12. *Under the assumption that the infinite line L is in a general position, we have the equality: $\Delta_f(t) = \Delta_C(t)$.*

Proof. Cf. [R], [D]. We see that $U \cong (\mathbf{C}^3 \backslash \{\tilde{f} = 0\}) \cap \{x_0 = 1\}$. The affine version of the Lefschetz theorem ([H]) asserts that $\pi_1(\mathbf{C}^3 \backslash \{\tilde{f} = 0\}) \to \pi_1(U)$ is an isomorphism. It follows that $H_1(\tilde{U}, \mathbf{Z}) \cong H_1(\tilde{F}, \mathbf{Z})$. Furthermore, the transformation T_* corresponds to $\tilde{h}_*$. Q.E.D.

Theorem 8. *Assume that L is in a general position. We have*

$$2q(X_n) = 1 + \#\{n\text{-th roots of unity which are roots of } \Delta_C(t)\} - B.C.$$

Corollary. *Under the same hypothesis, if $GCD(n, d) = 1$, then $q(X_n) = 0$.*

Proof. By hypothesis, we find that $b_1(U_n) = r - 1$ and $B.C. = r$.

We quote two divisibility theorems of the Alexander polynomials. See also [Ko], [LV].

Theroem 9 (Libgober [L]). *Suppose f is irreducible. Then*

$$\Delta_f(t) \mid \prod_{\tilde{p}} \Delta_{\tilde{p}}(t),$$

where $\tilde{p}$ moves all local branches of Sing*$(C \cup L)$.*

Theorem 10 (Dimca [D]). *Suppose f is reduced. Then*

$$\Delta_C(t) \mid \prod_{p \in \mathrm{Sing}\ (C)} \tilde{\Delta}_p(t),$$

where $\tilde{\Delta}_p(t)$ is the reduced local Alexander polynomial of p.

Corollary (Zariski [Z2]). *Suppose L is in a general position. If C has only nodes and ordinary cusps as its singularities, then $q(X_n) = 0$ unless $6|n$ and $6|d$.*

Proof. We know that $\Delta_p(t) = t - 1$ if p is a node, $= t^2 - t + 1$ if p is an ordinary cusp. Thus $\Delta_C(t) = (t-1)^{(r-1)}(t^2 - t + 1)^\ell$ for some ℓ. In view of Theorem 8, the assertion follows from this.

Remark. The assumption that L is in a general position is necessary in the above result. Let us consider the case:$f = (x + y)(x + y + 1)$. In this case, we find that $q(X_3) = 1$.

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